Radiation through a Plane-Parallel Absorbing Medium between Directional Surfaces

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Temperature distributions and heat flux are discussed for a gray gas in radiative equilibrium, bounded by directional walls maintained at different temperatures. The problem is solved with a differential approximation. This is a second approximation formulated to satisfy both the limits of isotropic and unidirectional radiation. Compatible boundary conditions are formulated on this same basis. Analytical results are obtained and compared to other calculations. The radiosity-normalized temperature function is generally not independent of wall properties even in the transparent limit, in contrast to the case of diffuse emission with either specular or diffuse reflection. The simple method which uses hemispherical emissivity and ignores reflection characteristics is evaluated. While generally not correct, this simplification leads to reasonable approximations in many cases.

I. Introduction

IT is possible to discern two fairly distinct avenues of research in radiative heat transfer with solid surfaces. One deals with heat exchange between surfaces through a nonparticipating medium. Here the concern is with effects of geometry and surface properties, and considerable progress has been made in accounting for real surfaces.[1–6] The other direction of work involves radiative transfer through an emitting and absorbing gas which is bounded by a solid wall and/or perhaps a shock wave, and the concern has been not so much with the wall as with the effects of the opacity of the radiatively participating medium. In the latter category one tends to find simple geometries and simple solid walls, either black or at most diffusely emitting and reflecting. While this distinction in approach is real, it is artificial with respect to real surfaces. The reason for assuming such simple surfaces for problems with participating media is, of course, the complexity of the problem.

The present paper deals with radiation through an absorbing gray medium in radiative equilibrium, bounded by infinite parallel solid gray walls maintained at different temperatures. The walls are allowed to emit and reflect nondiffusely, but they are restricted to be axisymmetric with respect to their normal direction. This situation was chosen because for diffuse walls it becomes a classical problem whose solution is well known. Through a normalization of the variables with radiosity, the diffuse problem becomes equivalent to that for black walls (see, for instance, Sparrow and Cess[7]). To obtain the solution for black walls is by no means a trivial accomplishment; several different schemes have been used (see again the text by Sparrow and Cess). The best is the solution of Heaslet and Warming,[8] expressed in terms of known tabulated functions. Some directional effects were found by Cess and Sotak[9] when one wall is allowed to become a specular reflector and diffuse emitter. These effects are weak, and they appear only at intermediate values of the optical depth between walls. Opaque and transparent radiosity-normalized heat flux and temperature were found to be the same as those for black walls. In the transparent limit all radiant energy leaving one surface must arrive at the other no matter how directionally distributed it may be. In the opaque limit all directional effects will be lost beyond a vanishingly small layer near the wall. Thus the limiting heat flux results of Ref. 9 will be generally true for all surfaces, even though the opaque limit is perhaps not too useful since the universal value for the heat flux is then small. It is tempting to argue generally that directional effects matter only at finite optical depths, since only then will attenuation be different for radiation leaving the surface normally compared to that leaving in a much more oblique direction. Hence the transparent and opaque normalized gas temperatures might also be thought to be generally independent of surface directional characteristics. The main purpose of the present study was to see what happens with directionally emitting axisymmetric surfaces. Much larger effects are found from those reported in Ref. 9. The radiosity-normalized temperature is found to depend on wall properties generally, even in the transparent limit. Thus the diffusely emitting surface is a very special case.

Because of the complexity of the problem for general surfaces even in the present simple geometry, it would not be possible to compute more than a few special cases from exact transfer theory without an extremely large computing effort. For this reason an approximate method is used in this study. The approximation used is a differential approximation. There exist several papers (Olfe,[10] Landram and Greif,[11] and Taitel[12]) in which it is claimed that a differential approximation just cannot properly take into account nonisotropic wall radiation. Indeed, it can be shown that the conventional first differential approximation gives the same results for directionally emitting walls, in terms of hemispherical emissivity, as for diffuse walls, failing to account in any way for reflection characteristics. The differential approximation used here, however, is a second approximation which allows considerably nonisotropy, since it is formulated to remain valid in the limit of transverse beam-like radiation. Such a method has led to some success in the far from isotropic problem of radiation between two concentric spheres,[13] where the usual differential approximation fails. A few checks, based on comparison of directional wall results from the present method with others, leads to the belief that the method can adequately handle the problem. It retains reflection characteristics. The method is quite simple, requiring only the integration of a fourth order ordinary differential equation with associated boundary conditions. General wall proper-

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ties enter these boundary conditions through a reflection coefficient and integrals of directional emissivity. Wall properties chosen in our examples are simple approximations to surface characteristics reported in the literature.[14–16] No attempt is made to incorporate any of the special surface models which have recently been developed, although this could easily be done. Simple analytical expressions are found for the heat flux in the opaque and transparent limits. The opaque expression has the form of a generalized slip coefficient.

Formulation of the model and boundary conditions is given for a plane parallel geometry without at first specializing to radiative equilibrium. That restriction is made subsequently to solve the present problem. The model is therefore also applicable to any plane-layer radiation problem with one or more isothermal directional walls, restricted only by the limitations imposed by grayness and local thermodynamic equilibrium.

II. Half-Range Moment Formulation

Because all the effects under investigation are introduced into the problem by the walls, special care is required to formulate boundary conditions. The most natural formulation for wall considerations in a plane-parallel geometry is with half-range moments. The present method provides the half-range counterpart to the full-range approach of Ref. 13. Half-range moments appear to be appropriate only for plane-parallel problems. They are defined with the help of Fig. 1. The radiation intensity I varies with direction. This is measured from the direction pointing to the left by the angle θ, with $\mu = \cos\theta$. The intensity is symmetric about the direction normal to the surfaces. The half-range moments measure separately radiation directed to the left and right in Fig. 1. Their sums and differences are then defined as follows:

$$I_n = 2\pi \int_0^1 I\mu^n d\mu + 2\pi \int_{-1}^0 I\mu^n d\mu$$
$$I^n = 2\pi \int_0^1 I\mu^n d\mu - 2\pi \int_{-1}^0 I\mu^n d\mu$$

Thus I_1 and I_0 are net heat flux (positive to the left) and average intensity, while the left-directed component of the heat flux from the right surface, or its radiosity, will be $(I_1 + I^1)_{\tau_0}/2$. The space coordinate τ is measured to the right from the left surface. It is an optical coordinate incorporating the absorption coefficient of the medium. The optical thickness of the layer is τ_0.

Assuming local thermodynamic equilibrium, the gray radiative transfer equation is

$$\mu\partial I(\mu,\tau)/\partial\tau = I - B, \quad B = \sigma T^4/\pi$$

Here B is the blackbody emissive power. One can multiply this equation by μ^n and then integrate with respect to direction. This can be done over all directions, leading to the conventional full-range moment equations. Alternatively, one can integrate separately over the ranges $-1 < \mu < 0$ and $0 < \mu < 1$. In that case, two equations result for each n. The result for the first four half-range equations is

$$\begin{aligned} dI_1/d\tau &= I_0 - 4\pi B, \quad dI^1/d\tau = I^0 \\ dI_2/d\tau &= I_1, \quad dI^2/d\tau = I^1 - 2\pi B \end{aligned} \tag{1}$$

Differential approximations result from an additional postulated algebraic relation between moments which closes the system and leads to an ordinary differential equation for some moment, conveniently chosen to be I_0. For instance, a semi-isotropic assumption ($I = I^+$ for $0 < \mu < 1$, $I = I^-$ for $-1 < \mu < 0$) leads to $I_1 = I^0/2$, $I^1 = I_0/2$. This, in conjunction with the first moment pair of Eqs. (1), leads to the well-known Schuster-Schwarzschild differential equation for the heat flux.

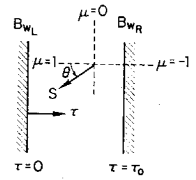

Fig. 1 Geometry of problem.

The present model is taken to be a higher approximation consisting of all of Eqs. (1) and, in addition, the following closing relations:

$$3I_2 - 4I^1 + I_0 = 0, \quad 3I^2 - 4I_1 + I^0 = 0 \tag{2}$$

These relations are rigorously true if the radiation is either isotropic or beamlike, whether directed to the right or left. The whole scheme is therefore built on the expectation that the model will not do too badly for some intermediate degree of nonisotropy.

Combining Eqs. (1) and (2) leads to

$$d^4I_0/d\tau^4 - 10d^2I_0/d\tau^2 + 9I_0 + 7(d^2/d\tau^2)(4\pi B) - 9(4\pi B) = 0 \tag{3}$$

This may be compared to the corresponding full-range equation, Eq. (11), of Ref. 13. As explained there, the heat flux can also be easily found in terms of I_0. The present form is

$$I_1 = (d/d\tau)[4\pi B/3 + (10/9)(I_0 - 4\pi B) - (1/9)d^2I_0/d\tau^2] \tag{4}$$

Equation (4) corresponds to Eq. (12) of Ref. 13. The difference in coefficients between full-range and half-range equations is similar to differences encountered between the equations of full-range and half-range spherical harmonics expansions.

Since Eq. (3) is fourth order, four boundary conditions need to be found. In the next section, two conditions are specified at each surface.

III. Boundary Conditions

Surface characteristics are determined by a directional emissivity $\epsilon(\mu)$ and a bidirectional reflectivity $\rho(\mu,\mu')$. Nomenclature and definitions follow Howell and Siegel.[15] For a gray surface, the directional emissivity equals the directional absorptivity. The directional emissivity is the ratio of the actual intensity emitted by the wall in some direction divided by the intensity B_w from a black surface at the wall temperature T_w. The quantity $\rho(\mu,\mu')$ is the reflected intensity in direction μ' divided by the contribution to the incident component of heat flux due to incident intensity in the direction μ. Finally, the angular-hemispherical reflectivity $\rho_{ah}(\mu)$ is the ratio of reflected heat flux integrated over all emergent directions divided by the incident heat flux contribution from incident intensity in the direction μ. It follows that for an axisymmetric surface on the right

$$\rho_{ah}(\mu) = 2\pi \int_0^1 \rho(\mu,\mu')\mu' d\mu' \tag{5}$$

For a nontransmitting surface

$$\rho_{ah}(\mu) = 1 - \epsilon(\mu) \tag{6}$$

Equation (6) is the result of equating the incident component of the heat flux to the difference between absorbed and reflected energy.

The emerging heat flux component is the sum of emitted and reflected energy. Thus, for the surface on the right in

Fig. 1, and letting the incident intensity be I^-,

$$2\pi \int_0^1 I\mu d\mu = 2\pi B_{w_R} \int_0^1 \epsilon_R \mu d\mu - 2\pi \int_{-1}^0 I^-(1 - \epsilon_R)\mu d\mu$$

Now the net heat flux I_1 is the sum of the half-range first moments, and one can write

$$I_1 = 2\pi B_{w_R} \int_0^1 \epsilon_R \mu d\mu + 2\pi \int_{-1}^0 I^- \epsilon_R \mu d\mu \quad (7)$$

Likewise, one finds for the difference

$$I^1 = 2\pi B_{w_R} \int_0^1 \epsilon_R \mu d\mu - 2\pi \int_{-1}^0 I^-(2 - \epsilon_R)\mu d\mu \quad (8)$$

Equations (7) and (8) provide a boundary condition for diffusely emitting surfaces. For if ϵ_R is independent of μ, then the unknown incident intensity I^- appears as a common integral in both equations which can be eliminated. Therefore one finds a relation between I_1 and I^1 involving only surface properties:

$$I_{1R} = [\epsilon_R/(2 - \epsilon_R)](2\pi B_{w_R} - I_R{}^1), \quad \epsilon_R = \text{const} \quad (9)$$

This relation is exact, and it is a consistent boundary condition for use with the half-range differential approximation. Notice that it is not necessary to specify reflection properties.

It is interesting to note what would happen for a directional surface if one were now to make the Schuster-Schwarzschild semi-isotropic assumption about the intensity. Then I^- becomes independent of μ, and again I^- can be eliminated. One finds the same relation between I_1 and I^1 just given, with ϵ_R replaced by $2\int_0^1 \epsilon_R(\mu)\mu d\mu$. This boundary condition is, of course, far from exact. If it is used with the first differential approximation one is led to the result that directional walls are approximated by diffuse walls with emissivity given by the directional emissivity integral above. This integral is just the hemispherical emissivity. Reflection characteristics are not accounted for. Constant ϵ applications of the semi-isotropic model can be found in a recent paper by Arpaci.[21]

Equations (7) and (8) are not useful as long as they contain the incident radiation field. Furthermore, they provide only one boundary condition. For the other, one needs to consider the half-range moments I_0 and I^0. Because these moments are not weighted by μ, one needs to define a similarly unweighted reflection property. Let $\rho_0(\mu)$ be the ratio of the reflected average intensity, integrated over all emergent directions, divided by the incident heat flux contribution from incident intensity in the direction μ. One finds, corresponding to Eq. (5),

$$\rho_0(\mu) = 2\pi \int_0^1 \rho(\mu,\mu')d\mu' \quad (10)$$

This reflection coefficient is related to $\epsilon(\mu)$ only through an integral restriction. Using the reciprocity property

$$\iint \mu\rho(\mu',\mu)d\mu' d\mu = \iint \mu' \rho(\mu,\mu')d\mu d\mu'$$

one obtains as an analog to Eq. (6):

$$\int_0^1 \rho_0(\mu)\mu d\mu = \int_0^1 (1 - \epsilon)d\mu \quad (11)$$

Consider now the emerging component of the average intensity for the surface on the right, again with incident intensity I^-:

$$2\pi \int_0^1 I d\mu = 2\pi B_{w_R} \int_0^1 \epsilon_R d\mu - 2\pi \int_{-1}^0 I^- \rho_{0_R} \mu d\mu$$

Then

$$I_0 = 2\pi B_{w_R} \int_0^1 \epsilon_R d\mu + 2\pi \int_{-1}^0 I^-(1 - \rho_{0_R}\mu)d\mu \quad (12)$$

$$I^0 = 2\pi B_{w_R} \int_0^1 \epsilon_R d\mu - 2\pi \int_{-1}^0 I^-(1 + \rho_{0_R}\mu)d\mu \quad (13)$$

Equations (12) and (13) are the counterpart of Eqs. (7) and (8), except that one can no longer eliminate the reflection coefficient and express everything in terms of emissivity. These relations supply an additional exact boundary condition in the special case of diffuse emission and diffuse or specular reflection, when the incident radiation field can again be eliminated.

One finds, respectively,

$$I_R{}^0 = \{4\pi B_w - I_0 - [4(1 - \epsilon)/\epsilon]I_1\}_R \quad \text{Diffuse Reflection} \quad (14a)$$

$$I_R{}^0 = \{[\epsilon/(2 - \epsilon)](4\pi B_w - I_0)\}_R \quad \text{Specular Reflection} \quad (14b)$$

This illustrates that the second boundary condition is influenced by differences in reflection characteristics.

As they stand, Eqs. (7, 8, 12, and 13) are exact but indeterminate for general surfaces. This situation is no different from that first encountered with the exact but indeterminate moment equations in Sec. II. The same kind of approximation will now be used to obtain two approximate boundary conditions.

Four new unknowns have appeared. These are the integrals:

$$i_1 = \int_{-1}^0 I^-\mu, \quad i_2 = \int_{-1}^0 I^-\mu d\mu$$

$$i_3 = \int_{-1}^0 I^-\rho_0\mu d\mu, \quad i_4 = \int_{-1}^0 I^-\epsilon\mu d\mu$$

One can find a linear relation between i_4, i_1, and i_2 which is satisfied when I^- is either isotropic, or a beam along $\mu = \pm 1$ for the left and right surfaces. A similar relation exists between i_3, i_1, and i_2. One now pretends that these relations are approximately true for any directional distribution of I^-. These two assumptions suffice to eliminate all the integrals containing I^- in two final relations between moments. For either wall these are of the following form:

$$I_{R,L}{}^1 = [\pm K_1 I_1 + I_0/2 - K_3(4\pi B_w - I_0)]_{R,L} \quad (15)$$

$$I_{R,L}{}^0 = [K_4 I_1 \pm K_6(4\pi B_w - I_0)]_{R,L} \quad (16)$$

The upper and lower sign is to be applied at the right and left walls, respectively.

The factors K_1, K_3, K_4, and K_6 contain all the wall properties. They are given below, using the notation

$$\epsilon_0 = \int_0^1 \epsilon d\mu_w, \quad \epsilon_1 = \int_0^1 \epsilon\mu_w d\mu_w$$

To make these integrals independent of surface orientation, surface properties are defined from here on with respect to the outward drawn normal, and μ_w is the direction cosine measured from that direction. The subscript $\perp$ refers to the outward normal direction.

$$K_1 = \frac{(1 + \rho_{0\perp})}{D}\left(\frac{2\epsilon_1}{\epsilon_\perp} - 1\right) - \left(\frac{2}{\epsilon_\perp} - 1\right)$$

$$K_3 = \frac{\epsilon_\perp}{D}\left(\frac{2\epsilon_1}{\epsilon_\perp} - 1\right) - \frac{1}{2}$$

$$K_4 = \frac{2(1 + \rho_{0\perp})}{D}\left(1 - \frac{\epsilon_1}{\epsilon_\perp}\right) + \frac{2}{\epsilon_\perp}, K_6 = \frac{2\epsilon_\perp}{D}\left(\frac{\epsilon_1}{\epsilon_\perp} - 1\right) - 1$$

$$D = (1 + \rho_{0\perp})\epsilon_1 - \epsilon_\perp(2 - \epsilon_0)$$

The occurrence of $\rho_{0\perp}$ and $\epsilon_\perp$ in these expressions originates in the beam limit which has been built into the approximation. This selects the normal direction as special. It is for radiation in this direction that interaction with a participating medium will be least effective in redistributing the intensity more uniformly over all directions, and a beam in this direction is correctly accounted for.

It may be useful to illustrate how some simple surfaces appear with this model. The associated K's are given in Table 1.

1) Diffuse emission, both specular and diffuse components of reflection:

$$\rho_{ah} = \rho_{ah}^{s} + \rho_{ah}^{d} = 1 - \epsilon,\ \rho_0 = \rho_0^{s} + \rho_0^{d} = \rho_{ah}^{s}/\mu + 2\rho_{ah}^{d}$$

$$\rho_{0\perp} = \rho_{ah}^{s} + 2\rho_{ah}^{d} = 1 - \epsilon + \rho_{ah}^{d}$$

2) Diffuse emission, diffuse reflection: $\rho_{0\perp} = 2(1-\epsilon)$
3) Diffuse emission, specular reflection: $\rho_{0\perp} = 1 - \epsilon$
4) Directional emission, diffuse reflection:

$$\rho_0(\mu) = 2[1 - \epsilon(\mu)],\quad \rho_{0\perp} = 2(1 - \epsilon_\perp)$$

$$2\epsilon_1 = \epsilon_0 \text{ [use Eq. (11)]}$$

Note that for cases 2 and 3 Eqs. (15) and (16) are identical to Eqs. (9) and (14), which are exact.

A combination of Eqs. (15) and (16) with Eqs. (1) allows the boundary conditions to be expressed in terms of I_0 alone. In this form they can be used directly with the differential equation of the present model, Eq. (3). The result is

$$\{(4K_4 - 3)d^2I_0/d\tau^2 \pm 12K_1 dI_0/d\tau + 12[K_3(4K_4 - 3) + 4K_6K_1](I_0 - 4\pi B_w) - 7(4K_4 - 3)(I_0 - 4\pi B)\}_{R,L} = 0 \quad (17)$$

$$\left\{(4K_4 - 3)d^3I_0/d\tau^3 - (40K_4 - 39)\frac{dI_0}{d\tau} + 7(4K_4 - 3)\frac{d(4\pi B)}{d\tau} \pm 36K_6(I_0 - 4\pi B_w)\right\}_{R,L} = 0 \quad (18)$$

For black walls ($K_1 = -1, K_3 = -\frac{1}{2}, K_4 = 0, K_6 = 1$) and a planar geometry, these become similar to the second approximation boundary conditions given in Ref. 13, with differences in coefficients attributable to the present half-range formulation.

IV. Radiative Equilibrium

So far, we have a general formulation for gray radiative transfer with local thermodynamic equilibrium in a planar geometry. It remains valid for any combination of radiation, conduction, and convection. We now specialize to purely radiative heat transfer. Consider the case of a participating medium in radiative equilibrium between two walls maintained steadily at different temperatures. Then $I_0 = 4\pi B$.

Equations (3, 17, and 18) become

$$d^4B/d\tau^4 - 3d^2B/d\tau^2 = 0 \quad (19)$$

$$\left\{\frac{4K_4 - 3}{12K_1}\frac{d^2B}{d\tau^2} \pm \frac{dB}{d\tau} + \frac{[K_3(4K_4 - 3) + 4K_6K_1}{K_1}(B - B_w)\right\}_{R,L} = 0 \quad (20)$$

$$\left\{-\frac{(4K_4 - 3)}{3(4K_4 - 6)}\frac{d^3B}{d\tau^3} + \frac{dB}{d\tau} \mp \frac{12K_6}{4K_4 - 6}(B - B_w)\right\}_{R,L} = 0 \quad (21)$$

Equation (19) integrates easily to give the emissive power distribution

$$B = A_1 \exp[(3)^{1/2}\tau] + A_2 \exp[-(3)^{1/2}\tau] + A_3\tau + A_4 \quad (22)$$

The heat flux for the present problem is a constant; it is given from Eq. (4) by

$$3I_1/4\pi = dB/d\tau - \tfrac{1}{3}d^3B/d\tau^3 = A_3 \quad (23)$$

Substitution of Eq. (22) into Eqs. (20) and (21) leads to four algebraic equations for the four integration constants. These equations will contain τ_0 and the wall properties as included in the K's, in addition to the two wall temperatures through B_{w_R} and B_{w_L}. For numerical results with general opacity it is convenient to normalize the wall temperatures out of the problem. This is done in terms of radiosity in the next section. The present form, however, suffices to give the transparent and opaque limits. After much tedious algebra one obtains eventually the following limiting results:

Table 1 Sample wall property coefficients

Case	K_1	K_3	K_4	K_6
1	$-\frac{(2-\epsilon)}{\epsilon}$	$-\frac{1}{2}$	$\frac{-4\rho_{ah}^{d}}{\epsilon(1+\rho_{ah}^{s})}$	$\frac{1-\rho_{ah}^{s}}{1+\rho_{ah}^{s}}$
2	$-\frac{(2-\epsilon)}{\epsilon}$	$-\frac{1}{2}$	$\frac{-4(1-\epsilon)}{\epsilon}$	1
3	$-\frac{(2-\epsilon)}{\epsilon}$	$-\frac{1}{2}$	0	$\frac{\epsilon}{2-\epsilon}$
4	$\frac{2(1-\epsilon_1)}{6\epsilon_1 - 4\epsilon_\perp}$	$\frac{\epsilon_1}{6\epsilon_1 - 4\epsilon_\perp}$	$\frac{8(\frac{1}{2} + \epsilon_1 - \epsilon_\perp)}{6\epsilon_1 - 4\epsilon_\perp}$	$\frac{-2\epsilon_1}{6\epsilon_1 - 4\epsilon_\perp}$

Transparent ($\tau_0 \to 0$):

$$q \equiv \frac{I_1}{\pi(B_{w_R} - B_{w_L})} = \frac{4(K_{3L}K_{6R} + K_{3R}K_{6L})}{(K_{6L} + K_{6R})(K_{1L} + K_{1R}) + (K_{3R} - K_{3L})(K_{4R} - K_{4L})} \quad (24)$$

$$\Theta \equiv \frac{B_{w_R} - B}{B_{w_R} - B_{w_L}} = \frac{K_{6L}(K_{1R} + K_{1L}) - K_{3L}(K_{4R} - K_{4L})}{(K_{6L} + K_{6R})(K_{1L} + K_{1R}) + (K_{3R} - K_{3L})(K_{4R} - K_{4L})} \quad (25)$$

Opaque ($\tau_0 \to \infty$):

$$q = 1/(G_R + G_L + \tfrac{3}{4}\tau_0) \quad (26)$$

with

$$G = [-(3)^{1/2} + \tfrac{3}{2} - K_4 + 6K_3 - 4(K_3K_4 + K_6K_1)]/4[K_6 - (3)^{1/2}K_3]$$

Note that opaque Eq. (26) has the form of the classical slip formula[17,18] generalized to directional walls. For a black surface, $G = 0.567$.

In the transparent limit the right sides of Eqs. (24) and (25) for two black surfaces become unity and $\frac{1}{2}$ respectively, as they should. In general the transparent heat flux will depend on wall properties. If the surfaces on both sides are the same, $\Theta = \frac{1}{2}$ and $q = 2K_3/K_1$. In the transparent limit the net heat flux becomes just the difference between the outward components of the heat flux leaving each surface, or the difference in surface radiosities. For this reason, a normalization with respect to radiosity is now introduced.

V. Radiosity Normalization

By definition, the radiosity of either surface is given by

$$R_R = 2\pi\int_0^1 I\mu d\mu = [(I^1 + I_1)/2]_R$$

$$R_L = -2\pi\int_{-1}^0 I\mu d\mu = [(I^1 - I_1)/2]_L$$

Consider the right surface: between Eqs. (7) and (8) one finds that in general

$$R_R = [2\pi\epsilon_1 B_w i_2/i_4 - (i_2/i_4 - 1)I_1]_R$$

This expression involves the incident flux I^- through the integrals i_2 and i_4. For diffuse emission one gets a result

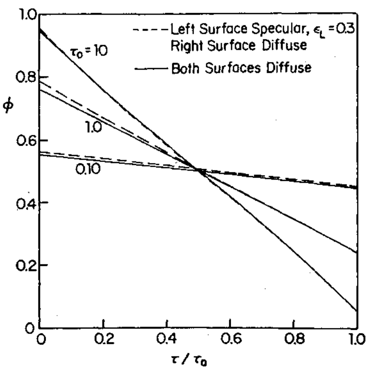

Fig. 2 Diffuse emission, present method.

which is independent of reflection properties, namely

$$R_R = \{\pi B_w - [(1 - \epsilon)/\epsilon]I_1\} \tag{27}$$

If one assumes I^- to be independent of direction, one finds this same expression except that $2\epsilon_1$ takes the place of ϵ. A less restrictive approximation, which will be used in the following, is obtained by a combination of Eqs. (7, 8, and 12), with $I_0 = 4\pi B$. This set of three equations contains all four of the i's, but by the assumption already made in Sec. III only two are independent. This again suffices to eliminate all integrals involving the incident intensity in a final expression involving only wall properties. The result, for either wall, is

$$R_{R,L} = \{\pi B_w \pm [(1 + K_1)/2]I_1 - (1 + 2K_3)\pi(B_w - B)\}_{R,L} \tag{28}$$

For a diffusely emitting surface this reduces to exact Eq. (27).

A radiosity normalized heat flux is now defined (see Ref. 7, but note the difference in sign convention) by $Q \equiv I_1/(R_L - R_R)$.

Since I_1 has been defined positive to the left, this means that, no matter what the surfaces, $Q = -1$ in the transparent limit. Correspondingly, a radiosity normalized temperature field is defined by $\phi \equiv (R_R - \pi B)/(R_R - R_L)$. Equations (19) and (22) expressed in ϕ remain unchanged. Equation (23) becomes

$$Q = \tfrac{4}{3}(d\phi/d\tau - \tfrac{1}{3}d^3\phi/d\tau^3) = \tfrac{4}{3}A_3$$

Introducing ϕ into Eqs. (20) and (21) results in the disappearance of wall temperature, since one can utilize the fact that

$$\begin{aligned}\phi_{wR} &= (1 + 1/2K_3)_R\phi_R + [(1 + K_1)/4K_3]_RQ\\ \phi_{wL} &= -1/2K_{3L} + (1 + 1/2K_3)_L\phi_L - [(1 + K_1)/4K_3]_LQ\end{aligned} \tag{29}$$

The resulting boundary conditions become

$$\begin{aligned}&[D_1d^2\phi/d\tau^2 + d\phi/d\tau + D_2\phi]_{\tau=\tau_0} = 0\\ &[D_1d^2\phi/d\tau^2 - d\phi/d\tau + D_2\phi]_{\tau=0} = 1\\ &[D_3d^3\phi/d\tau^3 + d\phi/d\tau + D_4\phi]_{\tau=\tau_0} = 0\\ &[D_3d^3\phi/d\tau^3 + d\phi/d\tau - D_4\phi]_{\tau=0} = 1\end{aligned} \tag{30}$$

Where

$$D_1 = -[K_3(4K_4 - 3) + 4K_6(1 + K_1)]/12K_3$$

$$D_2 = [K_3(4K_4 - 3) + 4K_6K_1]/2K_3$$

$$D_3 = [K_3(4K_4 - 3) + 4K_6(1 + K_1)]/3[K_3(4K_4 - 6) + 4K_6(1 + K_1)]$$

$$D_4 = 6K_6/[K_3(4K_4 - 6) + 4K_6(1 + K_1)]$$

It has already been mentioned that the use of radiosity for diffusely emitting and reflecting surfaces is rewarded by the complete disappearance of surface emissivities from the exact equations; that is, the normalization makes the problem equivalent to that with black walls. This property holds for the present approximation. For Case 2 of Sec. III the D's do not contain emissivity and they become identical to those for black walls, namely, $D_1 = \frac{1}{4}$, $D_2 = \frac{5}{2}$, $D_3 = -\frac{1}{6}$, $D_4 = 2$. However, wall emissivity does not disappear from the D's even in the simple case of diffuse emission, specular reflection. Nor should it, since this is just the directional wall effect described by Cess and Sotak.[9] The question remains whether ϕ in the transparent limit still retains wall properties in general. The transparent limit in the present variables also leads to a relatively simple analytical result as $\tau_0 \to 0$:

$$Q = -1, \quad \phi = \phi_R = \phi_L = \frac{1}{2}\left\{1 + \frac{[(K_1K_6 + K_3K_4)/K_3]_R - [(K_1K_6 + K_3K_4)/K_3]_L}{(K_6/K_3)_R + (K_6/K_3)_L}\right\} \tag{31}$$

The black value of ϕ is $\frac{1}{2}$. This will also be true for any pair of equal surfaces, but not otherwise except in special cases. It can be easily verified that pairs of surfaces, not equal but both belonging to the class given by Cases 1, 2, and 3 of Sec. III (diffuse emission), are such special cases.

It is a simple matter to substitute Eq. (22), written in ϕ, into Eqs. (30) to find the four integration constants for any given pair of walls as specified by their K's. For arbitrary τ_0 the expressions for the constants can be written out analytically, but they are fairly complicated. It is much easier to determine them numerically by solving the resulting four simultaneous algebraic equations on a computer. This then gives $\phi(\tau)$ and Q. All examples to be subsequently described were obtained in roughly three days' work, counting the programing and figure plotting.

From $\phi(\tau)$ and Q one needs to return to the physical variables $\Theta(\tau)$ and q defined in Sec. IV. This can be accomplished with the help of the relation

$$\psi = \frac{\pi(B_{wR} - B_{wL})}{R_R - R_L} = 1 - \left[\left(\frac{1 + K_1}{4K_3}\right)_L + \left(\frac{1 + K_1}{4K_3}\right)_R\right]Q + \left(1 + \frac{1}{2K_3}\right)_L(\phi_L - 1) - \left(1 + \frac{1}{2K_3}\right)_R\phi_R$$

Then

$$q = -Q/\psi$$

$$\Theta(\tau) = \frac{\phi(\tau) - (1 + 1/2K_3)_R\phi_R - [(1 + K_1)/4K_3]_RQ}{\psi}$$

In the transparent limit one conveniently has

$$\Theta = (\phi_{wR} - \phi)/(\phi_{wR} - \phi_{wL})$$

where ϕ_{wR} and ϕ_{wL} come from Eq. (29).

VI. Results and Discussion

Several examples with known exact results are considered first to establish confidence in the method. In the trans-

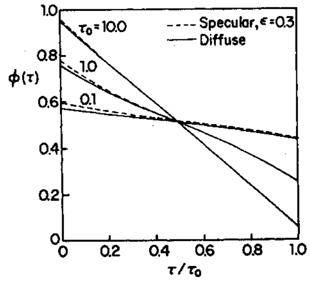

Fig. 3 Diffuse emission, exact.

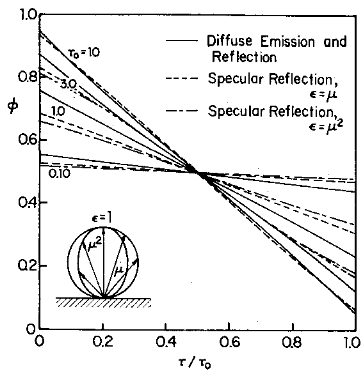

Fig. 4 Identical specular surfaces, dielectric directional emissivity.

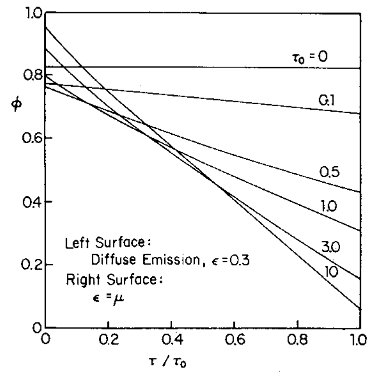

Fig. 6 Specular surfaces, dissimilar directional emissivities.

parent limit with diffusely emitting walls, reflection properties should have no influence on the net heat flux (Ref. 1, p. 110). This is verified with Eq. (24) and any pair of surfaces belonging to the class given by Case 1 of Sec. III. Another transparent example from Ref. 1, p. 164, places a directional surface opposite a black surface. The resulting heat flux is the same as for a diffusely emitting and reflecting surface, with $\epsilon = 2\epsilon_1$, opposite a black surface. We obtain a similar but more general result for any directionally emitting surface placed opposite a diffusely emitting and reflecting wall. Finally, a comparison is given between the results of the present method and those of Cess and Sotak[9] as reproduced by Sparrow and Cess.[7] Figure 2 shows the present results. These are to be compared with Fig. 3, which is a reproduction of Fig. 8-8 of Ref. 7. The present method shows some slight differences, with the greatest directional effect occurring at a larger τ_0. Essentially, however, agreement is good.

As already mentioned, $\phi = \frac{1}{2}$ for any pair of equal surfaces in the transparent limit. What happens for other opacities τ_0 is shown in Figs. 4 and 5 for specular surfaces.

Figure 4 exhibits directional effects typical of dielectric materials, with emissivity predominantly outwards rather than sideways. For each opacity there are two curves for progressively more directional surfaces, given by $\epsilon = \mu$ and $\epsilon = \mu^2$, and these are compared to the diffusely reflecting and emitting distribution. At small to intermediate τ_0 the curves are seen to have been rotated counterclockwise for predominantly outward emission. At large τ_0 the boundary-layerlike curvature of the profile near the wall has been reduced. Both effects resemble the action of an effective decrease in opacity. This is plausible on physical grounds, since predominantly outward radiation will have effectively a smaller mean free path than uniformly distributed radiation, for a given τ_0.

Figure 5 shows the reverse effect for predominantly sideways radiating surfaces typical of metals. The emissivity distribution here is given by

$$\epsilon(\mu) = \{1 + [(1 - \epsilon^{*2})/\epsilon^{*2}]\mu^2\}^{-1/2}$$

A polar plot of this emissivity is an ellipse stretched in a direction parallel to the surface, with $\epsilon(0) = 1$, $\epsilon(1) = \epsilon^*$. Results are shown for very nonuniform emissivities with $\epsilon^* = 0.04$ and 0.02. The distributions are rotated clockwise, and greater profile curvature appears near the wall for large τ_0. The situation for predominantly sideways radiation thus appears more opaque, as one might expect. Notice that ϕ is no longer bounded by zero and unity. This is unexpected only until one remembers that $\phi_{wR} = 0$, $\phi_{wL} = 1$ only for black walls, or in the opaque limit for all walls. In general the wall ordinates are located elsewhere, as determined by Eqs. (29), with $\phi_{wR} < 0$ and $\phi_{wL} > 1$.

Some nonsymmetrical results are shown in Figs. 6 and 7. Figure 6 illustrates what happens when a specular surface with uniform emissivity $\epsilon = 0.3$ is placed on the left, and a specular surface with $\epsilon = \mu$ is placed on the right. In the transparent limit ϕ is no longer $\frac{1}{2}$. The physical significance of this is that the radiosity from both sides is no longer weighted equally in the determination of the heating at a point in the medium. With increasing τ_0 the surface effects are progressively lost. Figure 7 is a more extreme example. Both sides are specular, on the right $\epsilon = \mu$ and on the left is the metallic-type surface used in Fig. 5 with $\epsilon^* = 0.04$. The transparent heating now weights the two walls very unevenly, favoring the wall on the right. Calculation of ϕ_{wR} and ϕ_{wL} shows that $(\phi - \phi_{wR}) < (\phi - \phi_{wL})$. This again makes sense, for it is radiation from that wall which suffers less interaction, and hence has a predominant influence in the interior.

The radiosity-normalized heat flux Q corresponding to these various cases is shown in Fig. 8. Surface directional effects appear only at intermediate opacities, and they are less spectacular here than for ϕ. The diffusely emitting cases agree well with what is given in Sparrow and Cess.[7] The case which has the least effective opacity is the symmetric, most normally emitting case of Fig. 4. The greatest impedance to the heat flux is offered by the symmetric, least normally

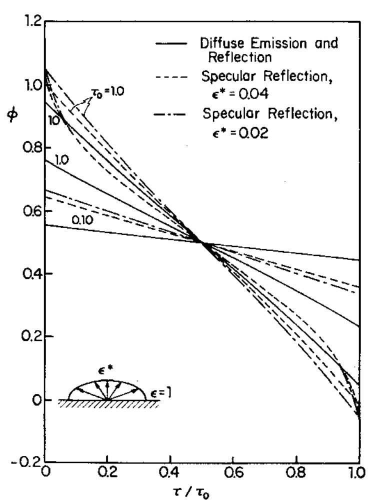

Fig. 5 Identical specular surfaces, metallic directional emissivity.

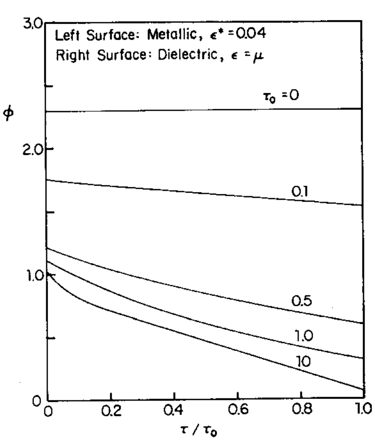

Fig. 7 Specular surfaces, dissimilar directional emissivities.

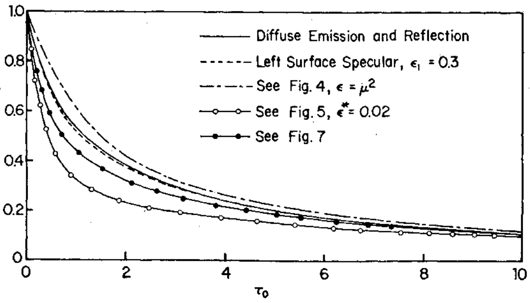

Fig. 8 Radiosity-normalized net heat flux.

emitting case in Fig. 5, that with $\epsilon^* = 0.02$. Opposing a metallic-type emissivity with a dielectric type leads to an intermediate heat flux not far from that for diffuse walls.

Figures 9 and 10 show the transformation from ϕ to Θ, and from Q to q, for the very unsymmetrical example of Fig. 7. The dashed curves are from a first approximation to be described in Sec. VII. The heat flux of Fig. 10 shows an interesting maximum at small τ_0. This is a directional effect due to the very small perpendicular emission of the left wall. Interaction with a participating medium both attenuates and tends to isotropize the intensity. In this example the latter effect serves to generate a directional distribution which is more effective for heat transfer, since the heat flux contains intensity weighted in favor of the normal direction. This counter-effect operates only with directional surfaces emitting predominantly sideways.

VII. First Approximation

The first, or semi-isotropic (Schuster-Schwarzschild) approximation for diffuse walls is obtained with Eq. (9) applied at both walls. The heat flux is found to be

$$q = 1/\{[(2-\epsilon)/2\epsilon]_R + [(2-\epsilon)/2\epsilon]_L + \tau_0\}$$

As already mentioned, the semi-isotropic model for directional walls leads to the same expression, with $2\epsilon_1$ replacing ϵ. If the variables are normalized with respect to radiosity, the problem becomes identical to that with black walls.

The rather remarkable result was noticed that in the transparent limit Eq. (24) reduced for certain special situations to the form

$$q = 1/\{[(1-\epsilon_1)/2\epsilon_1]_R + [(1-\epsilon_1)/2\epsilon_1]_L\} \tag{32}$$

These cases are pairs of surfaces, not necessarily equal but both belonging to the class given by Cases 1 and 4 of Sec. III, and also the case of any surface placed opposite a diffusely emitting and reflecting surface. Apparently some diffuse element, present on at least one side, is special. Although this simplification does not hold for general surfaces, it was nevertheless thought to be of interest to compare the flux from

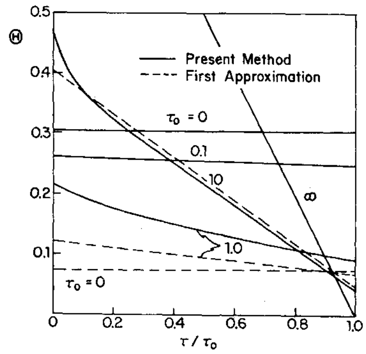

Fig. 9 Normalized emissive power for the example of Fig. 7.

Table 2 Transparent heat flux, first and present approximation

Walls	$q_{(24)}$	$q_{(32)}$
Symmetric, specular, $\epsilon = \mu$	0.60	0.50
Symmetric, specular, $\epsilon = \mu^2$	0.454	0.333
Symmetric, specular, elliptical ϵ with $\epsilon^* = 0.02$	0.0210	0.0204
Both specular, $\epsilon_R = \mu^2$, elliptical ϵ_L with $\epsilon^* =$ 0.02	0.0269	0.0196

Eq. (24) to that given by Eq. (32). Table 2 gives the results. Agreement is better than one might have expected.

Next, it will be of interest to see how closely the opaque limit as given by Eq. (26) contains wall properties in this same manner. For this purpose one can simply compare the quantity G with $(1-\epsilon_1)/2\epsilon_1$. Another useful comparison is between G and a surface property parameter contained in a slip formula recently obtained by Arpaci and Larsen.[19] This latter result is restricted to surfaces with diffuse emission and both specular and diffuse components of reflection. Table 3 shows the comparisons, and again agreement is quite good. It should be kept in mind that all results in Table 3 are approximate. The expression in Ref. 19 is in error by about 10% as compared to exact results.[20]

It is clear that, for the degree of directionality considered in the examples of this study, surface properties enter the denominator of the expressions for q in the transparent and opaque limits very roughly as $[(1-\epsilon_1)/2\epsilon_1]_R + [(1-\epsilon_1)/2\epsilon_1]_L$.

It is therefore suggested that, to an accuracy of perhaps 50% or better, the following first approximation may serve for initial heat transfer estimates for engineering surfaces:

$$q = 1/\{[(1-\epsilon_1)/2\epsilon_1]_R + [(1-\epsilon_1)/2\epsilon_1]_L + \tfrac{3}{4}\tau_0\}$$

This is the expression which was used to construct the dashed curve of Fig. 10. Notice that in that example it cannot account for the effect of directional redistribution of intensity leading to a heat flux maximum, but otherwise it does very well.

Since directional effects were found to be much larger on ϕ than on Q, one would not expect the first approximation to do very well in the prediction of Θ. The corresponding expression, likewise arbitrarily introducing a factor of $\frac{3}{4}$ into the optical coordinate, is

$$\Theta = \frac{[(1-\epsilon_1)/2\epsilon_1]_R + \frac{3}{4}(\tau_0 - \tau)}{[(1-\epsilon_1)/2\epsilon_1]_R + [(1-\epsilon_1)/2\epsilon_1]_L + \frac{3}{4}\tau_0}$$

This equation predicts linear profiles with various amounts of slip, and representative results are shown by the dashed curves in Fig. 9. Except for the larger values of τ_0, the first approximation is not useful for the prediction of temperature distributions with directional surfaces.

Table 3 Surface properties in slip formula, various approximations

Case	$G_{(26)}$	$\frac{(1-\epsilon_1)}{2\epsilon_1}$	Ref. (19)
Diffuse reflection, diffuse emission ϵ	$\frac{2-0.866\epsilon}{2\epsilon}$	$\frac{2-\epsilon}{2\epsilon}$	$\frac{2.125-\epsilon}{2\epsilon}$
Specular reflection diffuse emission ϵ	$\frac{2-\epsilon}{2\epsilon}\left(\frac{1+0.2215\epsilon}{1+0.0773\epsilon}\right)$	$\frac{2-\epsilon}{2\epsilon}$	$\frac{2-\epsilon}{1.778\epsilon}$
Specular reflection, $\epsilon = \mu$	0.845	1.00	
Specular reflection, $\epsilon = \mu^2$	1.222	1.50	
Specular reflection, elliptical ϵ with ϵ^* = 0.02	24.51	24.50	

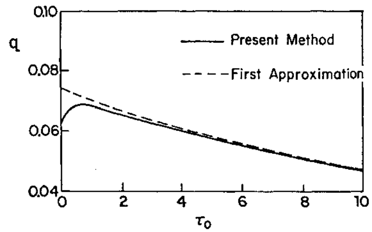

Fig. 10 The heat flux q for the example of Fig. 7.

VIII. Summary and Conclusions

A second differential approximation has been used to study directional wall effects on temperature distributions and heat flux in a participating gray medium in radiative equilibrium between infinite parallel gray walls. Results were found to be in agreement with the few results available from more exact methods. New results are physically plausible. Normalization of the variables with respect to radiosity minimizes the effect of directional wall properties on the heat flux, but this effect is still strongly contained in the temperature distributions, even in the transparent limit. The presence of at least one surface with diffuse emission and/or reflection leads to special simplifications, and conclusions based on experience with such surfaces are not generally valid. However, the heat flux obtained with the present method for some fairly directional wall examples was found to agree to within better than 50% with a first approximation, in which reflection properties are ignored and diffuse results are used, employing as emissivity the quantity

$$2 \int_0^1 \epsilon \mu_w d\mu_w$$

The method is simple, leading for the case of radiative equilibrium to a fourth order ordinary differential equation which can be integrated analytically. The formulation of the method has been more general than this, and a governing differential equation has been given, with appropriate boundary conditions, for radiation not restricted by radiative equilibrium.

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